Improved Nyquist Pulses

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***Abstract*—In this letter, we propose two new Nyquist (intersymbol interference free) pulses that exhibit better error probability performance in the presence of sampling errors than the popular raised-cosine and a recently proposed pulse by Beaulieu, Tan, and Damen. The new pulses are also robust to the root and truncation operations.**

***Index Terms*—Intersymbol interference (ISI), matched filters, pulse analysis.**

I. Known and Novel Nyquist Pulses

THE most commonly used Nyquist pulse is the raised-cosine pulse (rcos) whose frequency and impulse responses, respectively, read as [1]

$$S_1(f) = \begin{cases} 1, & |f| \le B(1-\alpha) \\ \frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{2B\alpha}(|f| - B(1-\alpha))\right), & B(1-\alpha) < |f| \le B(1+\alpha) \\ 0, & B(1+\alpha) < |f| \end{cases} \quad (1)$$

$$s_1(t) = \mathrm{sinc}(t/T)\frac{\cos(\pi\alpha t/T)}{1-(2\alpha t/T)^2} \quad (2)$$

where α is the roll-off factor, $B = 1/(2T)$ is the Nyquist frequency, and T is the transmission symbol period. The search of pulse shapes that yield no intersymbol interference (ISI) and low sensitivity to timing errors is a classical problem in data communications [2]. Recently, a new pulse has been proposed in [3]. We refer to it as *flipped-exponential* (fexp) pulse. Let $\beta = \ln 2/(\alpha B)$, then its frequency and impulse responses respectively read

$$S_2(f) = \begin{cases} 1, & |f| \le B(1-\alpha) \\ \exp(\beta(B(1-\alpha) - |f|)), & B(1-\alpha) < |f| \le B \\ 1 - \exp(\beta(|f| - B(1+\alpha))), & B < |f| \le B(1+\alpha) \\ 0, & B(1+\alpha) < |f| \end{cases} \quad (3)$$

$$s_2(t) = \frac{1}{T}\mathrm{sinc}(t/T)\frac{4\beta\pi t\sin(\pi\alpha t/T) + 2\beta^2\cos(\pi\alpha t/T) - \beta^2}{(2\pi t)^2 + \beta^2}. \quad (4)$$

Note that (3) is a corrected version of the one mistyped in [3].

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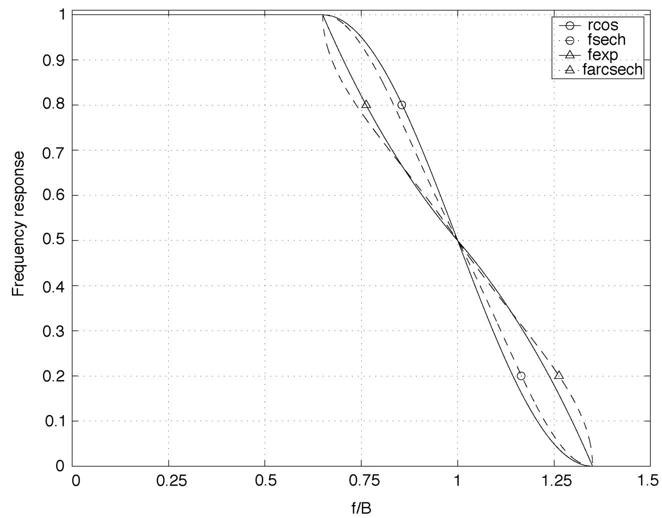

Fig. 1. Frequency responses of the considered pulses with roll-off factor $\alpha = 0.35$.

In this letter we propose two new Nyquist pulses. The first one that we refer to as *flipped-hyperbolic secant* (fsech) pulse, has frequency and impulse responses defined as

$$S_3(f) = \begin{cases} 1, & |f| \le B(1-\alpha) \\ \mathrm{sech}(\gamma(|f| - B(1-\alpha))), & B(1-\alpha) < |f| \le B \\ 1 - \mathrm{sech}(\gamma(B(1+\alpha) - |f|)), & B < |f| \le B(1+\alpha) \\ 0, & B(1+\alpha) < |f| \end{cases} \quad (5)$$

$$s_3(t) = \frac{1}{T}\mathrm{sinc}(t/T)\{8\pi t\sin(\pi\alpha t/T)F_1(t) + 2\cos(\pi\alpha t/T)[1 - 2F_2(t)] + 4F_3(t) - 1\} \quad (6)$$

where $\gamma = \ln(\sqrt{3}+2)/(\alpha B)$, and

$$F_1(t) = \sum_{k=0}^{+\infty}(-1)^k\frac{(2k+1)\gamma}{((2k+1)\gamma)^2 + (2\pi t)^2},$$
$$F_2(t) = \sum_{k=0}^{+\infty}(-1)^k\frac{(2\pi t)^2}{((2k+1)\gamma)^2 + (2\pi t)^2},$$
$$F_3(t) = \sum_{k=0}^{+\infty}(-1)^k\frac{(2\pi t)^2}{((2k+1)\gamma)^2 + (2\pi t)^2}e^{-(2k+1)\frac{\alpha\gamma}{2T}}. \quad (7)$$

The impulse response has been obtained through an expansion in series of exponentials.

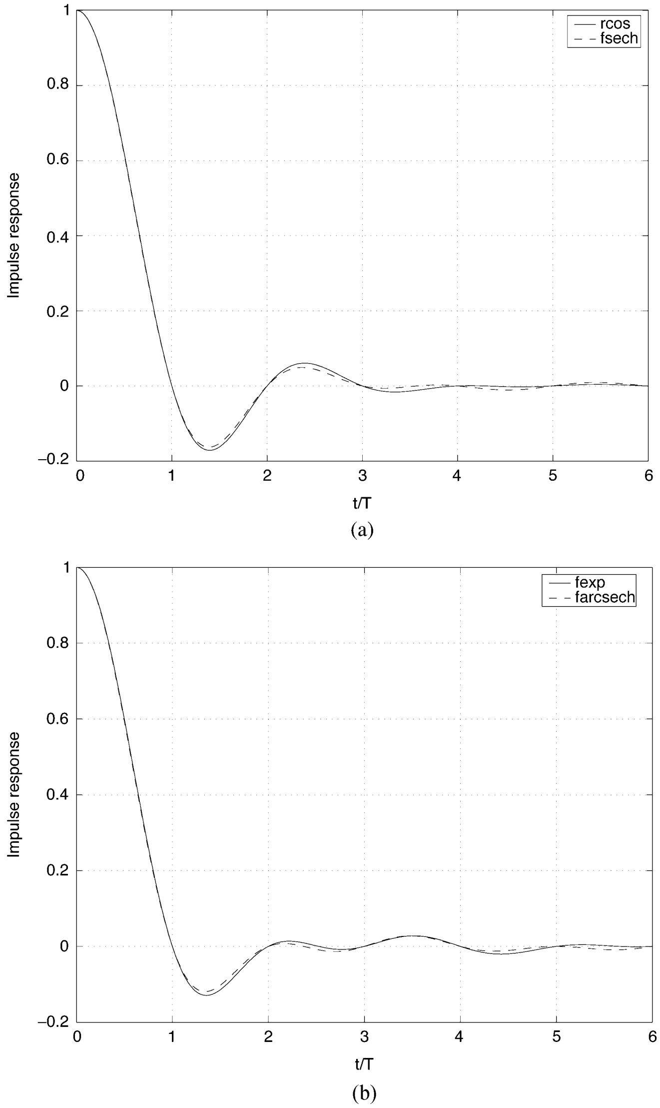

Fig. 2. Impulse responses with roll-off factor $\alpha = 0.35$: (a) Asymptotic decay as t^{-3}, (b) Asymptotic decay as t^{-2}.

The second pulse is referred to as *flipped-inverse hyperbolic secant* (farcsech) pulse. It has frequency response defined as

$$S_4(f) = \begin{cases} 1, & |f| \le B(1-\alpha) \\ 1 - \dfrac{1}{2\alpha B\gamma}\,\mathrm{arcsech}\left(\dfrac{1}{2\alpha B}(B(1+\alpha)-|f|)\right), & B(1-\alpha) < |f| \le B \\ \dfrac{1}{2\alpha B\gamma}\,\mathrm{arcsech}\left(\dfrac{1}{2\alpha B}(|f|-B(1-\alpha))\right), & B < |f| \le B(1+\alpha) \\ 0, & B(1+\alpha) < |f|. \end{cases} \tag{8}$$

Its impulse response $s_4(t)$ can be evaluated through a numerical inverse Fourier transform.

We plot the frequency and impulse responses of the above pulses in Figs. 1 and 2. Note that the pulses are real, even, and ISI free with optimum time sampling. Further, $s_1(t)$, and $s_3(t)$ asymptotically decay as t^{-3}, while $s_2(t)$, and $s_4(t)$ as t^{-2}. However, it is interesting to note that the amplitude of the main sidelobes is smaller for the pulses with t^{-2} decay than for those with t^{-3} decay.

TABLE I
BIT ERROR PROBABILITY FOR $N = 2^9$ INTERFERING SYMBOLS AND $\mathrm{SNR} = 15$ dB. $\mathrm{BER} = 9.3610\mathrm{e}{-9}$ AT THE OPTIMUM SAMPLING TIME

α	pulse	$t/T = \pm 0.05$	$t/T = \pm 0.1$	$t/T = \pm 0.2$	$t/T = \pm 0.3$
0.25	rcos	8.2189e-8	2.8184e-6	9.7462e-4	2.5914e-2
	fsech	7.5579e-8	2.3337e-6	7.7201e-4	2.2982e-2
	fexp	5.8117e-8	1.2980e-6	3.5678e-4	1.4524e-2
	farcsech	5.3996e-8	1.1011e-6	2.8405e-4	1.2496e-2
0.35	rcos	5.9997e-8	1.3896e-6	3.9084e-4	1.5481e-2
	fsech	5.4002e-8	1.0944e-6	2.8000e-4	1.2471e-2
	fexp	3.9253e-8	5.4021e-7	1.0129e-4	5.8880e-3
	farcsech	3.5970e-8	4.4580e-7	7.6203e-5	4.6950e-3
0.5	rcos	3.9723e-8	5.4890e-7	1.0217e-4	6.0067e-3
	fsech	3.4949e-8	4.1186e-7	6.6009e-5	4.2284e-3
	fexp	2.4134e-8	1.8580e-7	2.0878e-5	1.5772e-3
	farcsech	2.1875e-8	1.4916e-7	1.5344e-5	1.2253e-3

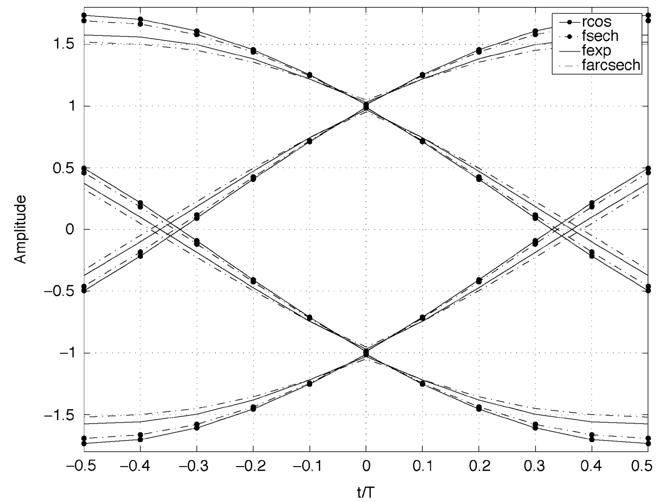

Fig. 3. Eye diagram for the equivalent truncated, in $[-5.5T, 5.5T]$, root-pulses auto-convolution. Only inner–outer contours are shown for $\alpha = 0.35$.

In Table I we report the bit error rate (BER) in the presence of time sampling errors computed according to [4], assuming BPSK signaling. For all different values of roll-off factor and sampling time errors that we consider, the t^{-2} pulses beat the t^{-3} pulses. In particular, the proposed fsech pulse is better than the rcos pulse, likewise the proposed farcsech pulse is better than the fexp pulse.

II. Truncated Root-Pulse Behavior

It is common engineering practice to split the pulse in half, i.e., to deploy a root (of the frequency response) pulse at the transmitter side, and the same root pulse at the receiver side that acts as a matched filter [5]. Further, the filters are digitally implemented with a truncated oversampled causal version. For complexity reasons it is desirable that the tails decay rapidly in order to allow quick truncation, and minimize the resulting intersymbol interference (ISI) at the receiver. This is because the auto-convolution of the truncated root pulse is not Nyquist anymore. We consider this aspect in Fig. 3, where we plot the

TABLE II
BIT ERROR PROBABILITY FOR THE TRUNCATED IN, $[-5.5T, 5.5T]$, VERSIONS OF THE PULSES, $\mathrm{SNR} = 15$ dB

α	pulse	$t/T = 0$	$t/T = \pm 0.05$	$t/T = \pm 0.1$	$t/T = \pm 0.2$
0.25	rcos	9.7458e-9	8.5408e-8	2.9203e-6	9.9832e-4
	fsech	1.0466e-8	8.3109e-8	2.5055e-6	7.9781e-4
	fexp	1.2057e-8	7.2516e-8	1.5433e-6	3.9005e-4
	farcsech	1.1468e-8	6.5543e-8	1.3128e-6	3.2160e-4
0.35	rcos	9.4705e-9	6.0754e-8	1.4087e-6	3.9563e-4
	fsech	9.5417e-9	5.5189e-8	1.1233e-6	2.8756e-4
	fexp	9.7702e-9	4.1449e-8	5.7905e-7	1.0960e-4
	farcsech	1.0593e-8	4.0722e-8	4.9870e-7	8.2087e-5
0.5	rcos	9.4158e-9	3.9954e-8	5.5192e-7	1.0259e-4
	fsech	9.4404e-9	3.5252e-8	4.1541e-7	6.6490e-5
	fexp	9.4443e-9	2.4637e-8	1.9341e-7	2.2267e-5
	farcsech	9.6516e-9	2.2914e-8	1.5992e-7	1.6772e-5

eye diagrams that correspond to the auto-convolution of the root-pulses truncated in $[-5.5T, 5.5T]$. Only the relevant contours of the eye diagrams are reported, i.e., the curves generated by the two sequences that yield the maximum and minimum ISI value [6]. Note that closed-form expressions for the root-pulse responses exist only for the rcos pulse. As the figure shows, even with optimum time sampling, ISI arises, and it is more pronounced for the pulses with t^{-2} decay. In general, it is found that the pulses with t^{-3} decay have better behavior for very small sampling errors. On the contrary, for sampling offsets larger than $0.05T$ the two pulses that decay as t^{-2} exhibit a larger noise margin. In particular, the farcsech pulse has the largest eye opening as well as the minimum maximum distortion. The behavior predicted by the eye diagram of Fig. 3 is confirmed by Table II that reports BER for various roll-factors and sampling errors.

III. CONCLUSION

We have proposed two new Nyquist pulses that asymptotically decay respectively as t^{-3} and as t^{-2}. They both exhibit a smaller error probability than the popular raised-cosine pulse. The latter pulse behaves also better than a recently proposed pulse by Beaulieu, *et al.* [3]. Next, we have considered the equivalent pulses that are obtained by the auto-convolution of a "truncated root version". For very small timing errors, the t^{-3} decay pulses exhibit better performance than the t^{-2} decay pulses. For larger timing errors the proposed t^{-2} pulse still exhibits a much wider eye opening, a smaller maximum distortion and improved BER than the other pulses.

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