Analysis of the Achievable Time-Frequency Diversity Gains in Coded OFDM

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Abstract — We address the problem of determining how to optimally exploit the time-frequency diversity in a coded OFDM system. We assume to deploy a code that generates complex codewords with M-ary alphabet of length equal to NN_B where N is the number of tones. The codewords span N_B blocks over which the fading is assumed to be quasi-static but independent across distinct blocks. We determine rate/complexity tradeoffs for maximum achievable diversity gains, and coding gains as a function of the channel response. We dig into the benefits and disadvantages of deploying interleaving, and we show that it helps to maximize diversity/coding gains with high probability.

Keywords — Coding, diversity, fading channels, interleaving, OFDM.

1. Introduction

Wideband channels exhibit frequency selectivity and consequently severe time dispersion. When transmission is over such channels the receiver must implement some form of equalization to overcome the intersymbol interference introduced by the time-dispersive channel. The equalization task simplifies when transmission is based on some form of multicarrier modulation. Orthogonal frequency division multiplexing (OFDM) has become a popular multicarrier modulation technique that allows for a very simple detection scheme [9]. It is based on deploying, at the transmitter side, an N-point IFFT followed by the insertion of a cyclic prefix. The receiver simplifies to a block that disregards the cyclic prefix, followed by an N-point FFT. If the cyclic prefix is longer than the channel time dispersion, at each FFT output we get the data symbol transmitted on that sub-channel weighted by the channel frequency response, plus thermal noise. Therefore, the detector simplifies to a one-tap equalizer, i.e., a symbol decision device that requires only knowledge of the sub-channel weight.

If transmission is uncoded, the frequency diversity provided by the frequency selective fading channel cannot be exploited. In other words, if a sub-carrier frequency coincides with a channel null, then the information carried by that sub-carrier is lost. Therefore, channel coding has to be deployed in order to exploit some frequency diversity, and grant reliable transmission. The resulting scheme is often referred to as coded OFDM (COFDM) [3],[9],[10].

In this paper we address the problem of determining how to optimally exploit the time and frequency diversity in a coded OFDM system. We assume to deploy a channel encoder before OFDM modulation. The encoder generates complex codewords, with M-ary alphabet, of length equal to NN_B where N is the number of tones. The channel is assumed to be time-variant and frequency selective. In particular we assume a block Rayleigh fading model, i.e., the channel is assumed to be time-invariant over the duration of a given block but it varies in an independent fashion in the adjacent block. The codewords span one or more blocks. The cyclic prefix is assumed to be long enough to cope with the channel time dispersion.

The deployment of coding has the potentiality of yielding a diversity and a coding gain over the uncoded OFDM system. We study the error rate performance, and we derive design criteria for COFDM such that both the diversity gain, and the coding gain are maximized. The analysis goes through the analysis of the pairwise error probability. In turn the pairwise error probability depends on the rank and determinant of a generalized Vandermonde matrix whose structure is a function of the channel and the error event itself. We derive necessary and sufficient conditions under which the generalized Vandermonde matrix is full rank. Not only the Hamming distance but also the error position patterns play an import role. It is found that certain codes with Hamming distance equal to d (with d being the rank of the channel correlation matrix) have pathologic behavior and do not achieve full diversity when used in a conjunction with OFDM modulation.

We also consider the deployment of time-frequency interleaving. The idea is widely applied, however, the approach is often heuristic. Instead, our analysis helps understanding the effect of the deployment of interleaving in terms of coding and diversity gains. It is found that interleaving may decrease the diversity exploitation of the code. However, it makes it unlikely. In certain conditions, interleaving can preserve the diversity gain of the code but it is beneficial since it is capable of increasing the coding gain.

2. Coded OFDM

We consider a coded OFDM system architecture as described in what follows. An information bit data block is channel encoded into an M-ary complex codeword $\mathbf{a}$ of length NN_B. The codeword is split into N_B sub-words of length N, $\{a^k(l)\}$, $k = 0,...,N-1$, $l = 0,...,N_B-1$. Each sub-word is OFDM modulated by applying an N-point IDFT whose outputs are

$$x^n(l) = \sum_{k=0}^{N-1} a^k(l) e^{j\frac{2\pi}{N}kn} \quad . \tag{1}$$

A cyclic prefix of length μ symbols is added to cope with the time dispersion introduced by the channel. We emphasize that $a^k(l)$ is the encoded complex data symbol transmitted on sub-channel k over block l at rate $1/T_0$ with $T_0 = (N+\mu)T$. It belongs, for instance, to the M-QAM constellation.

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After P/S conversion, the symbols $x(nT+lT_0)=x^n(l)$ are transmitted over a frequency selective fading channel. In our baseband discrete time model, we assume a channel impulse response $g(nT;lT_0)=\sum_{i=1}^{N_P}\alpha_i(l)\delta(nT-p_iT)$ with $p_i \subseteq \{0,...,\mu\}$ being the $N_P \le \mu$ tap delays. Note that the channel can be sparse. The channel tap gains are assumed to be complex Gaussian with zero mean (Rayleigh fading), and normalized power $\sum_i E[|\alpha_i|^2]=1$. The channel taps are time-invariant over the duration of an OFDM symbol, however, they may vary over the following block in an independent fashion. This model corresponds to the well known block fading model [4].

The receiver comprises conventional OFDM demodulation followed by maximum likelihood channel decoding. After having disregarded the cyclic prefix, the *N*-point DFT outputs read

$$z^k(l)=a^k(l)H^k(l)+w^k(l) \qquad H^k(l)=\sum_{i=1}^{N_P}\alpha_i(l)e^{-j\frac{2\pi}{N}kp_i} \tag{2}$$

for $k=0,...,N-1$, where $w^k(l)$ is a sequence of i.i.d. Gaussian random variables with zero mean, and variance N_0. Under the AWGN assumption, the optimal maximum likelihood channel decoder decides in favor of the codeword $\hat{\mathbf{a}}=\{\hat{a}^k(l)\}_{l=0,...,N_B-1}^{k=0,...,N-1}$ that maximizes the Euclidean distance metric $\Delta=\sum_{l=0}^{N_B-1}\sum_{k=0}^{N-1}|z^k(l)-H^k(l)\hat{a}^k(l)|^2$.

3. Pairwise Error Probability

The pairwise error probability conditioned on the channel state information is

$$P(\mathbf{a}\to\hat{\mathbf{a}}\,|\,\mathbf{H})=Q\left(\sqrt{\frac{d^2(\mathbf{a},\hat{\mathbf{a}})}{2N_0}}\right)\le\frac{1}{2}e^{-\frac{d^2(\mathbf{a},\hat{\mathbf{a}})}{4N_0}}. \tag{3}$$

The right most inequality follows from the conventional Chernoff bound, and the squared pairwise error distance between the transmitted codeword and the codeword we decide in favor of, is defined as

$$d^2(\mathbf{a},\hat{\mathbf{a}})=\sum_{l=0}^{N_B-1}\sum_{k=0}^{N-1}|H^k(l)|^2|\varepsilon^k(l)|^2 \tag{4}$$

with $\varepsilon^k(l)=a^k(l)-\hat{a}^k(l)$. From (2) we can write that

$$d^2(\mathbf{a},\hat{\mathbf{a}})=\sum_{l=0}^{N_B-1}\sum_{m,n=1}^{N_P}\alpha_m(l)\alpha_n^*(l)\sum_{k=0}^{N-1}e^{j\frac{2\pi}{N}k(p_n-p_m)}|\varepsilon^k(l)|^2. \tag{5}$$

If we use matrix notation with $\boldsymbol{\alpha}(l)=[\alpha_1(l),....,\alpha_{N_P}(l)]^T$, and $\mathbf{G}(l)$ being the $N_P\times N_P$ matrix whose elements are $G_{n,m}(l)=\sum_{k=0}^{N-1}e^{j\frac{2\pi}{N}k(p_n-p_m)}|\varepsilon^k(l)|^2$, we can rewrite (5) as a normal quadratic form [5],[8]

$$d^2(\mathbf{a},\hat{\mathbf{a}})=\sum_{l=0}^{N_B-1}\boldsymbol{\alpha}^H(l)\mathbf{G}(l)\boldsymbol{\alpha}(l)=\sum_{l=0}^{N_B-1}\sum_{m=0}^{N_P-1}\lambda_{m,l}|\beta_{m,l}|^2. \tag{6}$$

In (6) $\lambda_{m,l}$ $m=0,...,N_P-1$ are the N_P eigenvalues of the matrix $\mathbf{R}(l)\mathbf{G}(l)$, with $\mathbf{R}(l)=E[\boldsymbol{\alpha}(l)\boldsymbol{\alpha}^H(l)]$ being the channel taps correlation matrix (assumed to be positive definite), while $\beta_{m,l}$ is a sequence of i.i.d. Gaussian random variables with zero mean and unit variance. Note that in our model the channel tap gains are time-invariant over a single OFDM symbol but change independently in the adjacent block. Therefore we can drop the dependency of the index l in the channel taps correlation matrix.

Under the Rayleigh fading assumption the probability density function of the squared distance can be evaluated in closed-form [5],[8], which yields

$$p_{d^2}(a)=\sum_{i=1}^{N'}\sum_{n=1}^{m_i}A_{i,n}\frac{a^{n-1}}{\lambda_i^n(n-1)!}e^{-\frac{a}{\lambda_i}}1(a) \tag{7}$$

with $\lambda_i=\lambda_{m,l}$ for $i=lN_P+m$, and $N'\le N_P$ is the number of distinct eigenvalues each with multiplicity $m_1,...,m_{N'}$. Further, $A_{i,n}$ are the coefficients of the partial fraction expansion of the characteristic function (residues)

$$A_{i,n}=\frac{1}{(-\lambda_i)^{m_i-n}(m_i-n)!}\left[\frac{d^{m_i-n}}{ds^{m_i-n}}\left\{\prod_{p=1,p\ne i}^{N'}\left(1-\lambda_p s\right)^{-m_p}\right\}\right]_{s=1/\lambda_i}. \tag{8}$$

If we average over (7) we obtain the average pairwise error probability in closed-form

$$P(\mathbf{a}\to\hat{\mathbf{a}})=\frac{1}{2}\sum_{i=1}^{N'}\sum_{n=1}^{m_i}A_{i,n}\left[1-\sum_{l=0}^{n-1}\frac{(2l)!}{2^{2l}(l!)^2}\sqrt{\frac{\lambda_i/4N_0}{(1+\lambda_i/4N_0)^{2l+1}}}\right]. \tag{9}$$

Finally, with the Union Bounding technique we can evaluate the average bit and frame error rates.

3.1 Chernoff Bound and Design Criteria

Averaging over the distribution of $|\beta_{m,l}|^2$ (exponential with Rayleigh fading) the right most term in (3) yields

$$P(\mathbf{a}\to\hat{\mathbf{a}})\le\frac{1}{2}\prod_{l=0}^{N_B-1}\prod_{m=0}^{N_P-1}\left(1+\frac{\lambda_{m,l}}{4N_0}\right)^{-1}\le\frac{1}{2}\left(\frac{E_S}{4N_0}\right)^{-r}\prod_{\lambda_{m,l}\ne0}\left(\frac{\lambda_{m,l}}{E_S}\right)^{-1} \tag{10}$$

where $r=\sum_{l=0}^{N_B-1}r_l$, and r_l equals the number of non-zero eigenvalues of $\mathbf{RG}(l)$. Therefore, the COFDM system can achieve a diversity gain (slope of the error rate curve in the log scale) of *r*, and a coding gain (shift of the error rate curve) of $\prod_{\lambda_{m,l}\ne0}\left(\lambda_{m,l}/E_S\right)$ with $E_S=E[|a^k(l)|^2]$. The design of the code should maximize the diversity and coding gains [4],[6].

Note that if the channel is flat there is no frequency diversity available. However, maximum time diversity exploitation can be obtained with coding across N_B blocks if the code has minimum time Hamming distance d_{TH} (number of non-zero $d_{TE}(l)=\sum_{k=0}^{N-1}|\varepsilon^k(l)|^2$ for $l=0,...,N_{B-1}$) equal to N_B [4]. On the other hand, if the channel is quasi-static, e.g., when coding is limited to a single block, the only source of diversity is the frequency diversity. In the remainder of this paper we mostly focus on the quasi-static channel case.

4. Quasi-Static Frequency Selective Fading Channel

Let us assume a quasi-static frequency selective channel with full rank correlation matrix $d=rank(\mathbf{R})=N_P<N$. Without loss of generality we can assume coding over a single OFDM symbol (block) and set $l=0$. Then, the (frequency) diversity gain is $r=rank(\mathbf{G})\le d$. The CODFM fully exploits the channel diversity only when $rank(\mathbf{G})=d$. In such a case,

$$P(\mathbf{a}\to\hat{\mathbf{a}})\le0.5(4N_0)^{-d}|\mathbf{R}|^{-1}|\mathbf{G}|^{-1} \tag{11}$$

and consequently the maximization of the coding gain calls for the maximization of the determinant of the matrix $\mathbf{G}$. If we assume an error sequence of length L we can explicitly write $\mathbf{G}$ as follows:

$$\mathbf{G} = \mathbf{F}^H \mathbf{E} \mathbf{F} \qquad \mathbf{E} = \text{diag}\ [|\varepsilon^{k_1}|^2, ..., |\varepsilon^{k_L}|^2]$$

$$\mathbf{F} = \begin{bmatrix} \omega^{-k_1 p_1} & e^{-k_1 p_2} & ... & \omega^{-k_1 p_d} \\ \omega^{-k_2 p_1} & \omega^{-k_2 p_2} & ... & ... \\ ... & ... & ... & ... \\ \omega^{-k_L p_1} & \omega^{-k_L p_2} & ... & \omega^{-k_L p_d} \end{bmatrix} \quad \omega = e^{j\frac{2\pi}{N}} \tag{12}$$

where the error positions are assumed to be $0 \le k_1 < k_2 < ... < k_L < N$, and the tap delays are assumed to be $0 \le p_1 < p_2 < ... < p_d < N$.

Let $\mathbf{E} = \mathbf{E}_1^* \mathbf{E}_1$, then $\mathbf{G} = (\mathbf{E}_1 \mathbf{F})^H \mathbf{E}_1 \mathbf{F}$. It follows that $rank(\mathbf{G}) = rank(\mathbf{E}_1 \mathbf{F}) = rank(\mathbf{F})$ since $\mathbf{E}_1$ is full rank equal to L. If we define the relative error distances $\delta_i = k_i - k_1$, $i = 1,...,L$, and the relative channel delays $\tau_i = p_i - p_1$, $i = 1,...d$, we can write

$$\mathbf{F}^H = \mathbf{F}_1 \tilde{\mathbf{F}} \mathbf{F}_2$$

$$\mathbf{F}_1 = \text{diag}\ [1, \omega^{k_1 \tau_2}, ..., \omega^{k_1 \tau_d}] \quad \mathbf{F}_2 = \text{diag}\ [\omega^{k_1 p_1}, ..., \omega^{k_L p_1}] \tag{13}$$

$$\tilde{\mathbf{F}} = \begin{bmatrix} 1 & 1 & ... & 1 \\ 1 & \omega^{\tau_2 \delta_2} & ... & \omega^{\tau_2 \delta_L} \\ ... & ... & ... & ... \\ 1 & \omega^{\tau_d \delta_2} & ... & \omega^{\tau_d \delta_L} \end{bmatrix}.$$

Therefore, the rank of $\mathbf{F}$ coincides with the rank of $\tilde{\mathbf{F}}$. $\tilde{\mathbf{F}}$ is obtained by deleting some rows and some columns from an $N \times N$ Vandermonde matrix $\mathbf{W}$ [2]. From the above results we can state the following two propositions.

Proposition 1. In quasi-static Rayleigh fading, to achieve a frequency diversity gain of d (full diversity) the Hamming weight L of the error sequence has to be at least d. Consequently, to achieve full frequency diversity the minimum frequency Hamming distance d_{FH} (number of nonzero $d_{FE}(k) = |\varepsilon^k (l = 0)|^2$ for $k = 0,..., N-1$) of the code has to be at least d.

Proposition 2. The maximum achievable diversity with a code having minimum Hamming distance d_{FH} is $\min\{d_{FH}, d\}$.

Therefore, in quasi-static frequency selective fading, the diversity gain is determined not only by the frequency Hamming weight of the error events, but also by the error patterns (positions). The shape of the constellation is irrelevant with respect to the code diversity.

In the following we state several necessary and/or sufficient conditions under which $rank(\mathbf{G}) = rank(\mathbf{F}) = rank(\tilde{\mathbf{F}}) = r$.

Proposition 3. All error sequences of length N achieve full diversity, i.e., $rank(\mathbf{G}) = d$.

Proof. When the length of the error sequence is N the matrix $\tilde{\mathbf{F}}$ is obtained from a $N \times N$ Vandermonde matrix $\mathbf{W}$ where a number of rows has been deleted. Since we assume $0 \le p_1 < ... < p_d < N$, the $d \times N$ matrix $\tilde{\mathbf{F}}$ is full rank.

Proposition 4. All error sequences of length one do not achieve any diversity benefit. Consequently, uncoded OFDM does not provide any diversity gain.

Proof. We just need to notice that an error sequence of length one generates the following Toeplitz-Hermitian matrix

$$\mathbf{G} = |\varepsilon^{k_1}|^2\ Toeplitz\{1, \omega^{k_1(p_1 - p_2)}, ..., \omega^{k_1(p_1 - p_d)}\}. \tag{14}$$

Such a matrix has rank equal to one, so that $\mathbf{RG}$ has also rank one with the only non-zero eigenvalue $\lambda(k_1) = trace\{\mathbf{RG}\}$. The fact that the eigenvalue is a function of the error event position translates into sub-channels with different error rates whenever the channel taps are correlated [8].

Proposition 5. If the channel has rank N, the diversity order equals the (frequency) Hamming weight of the error sequence. In other words, the minimum Hamming distance of the code determines the diversity order.

Proof. It can be proved similarly to Proposition 3.

Proposition 6. If the relative channel delays, or the relative error distances are consecutively ordered respectively as $\tau_i = i - 1$, $i = 1,2,3,...d$, and $\delta_i = i - 1$, $i = 1,2,3,..., L$, the diversity gain is $\min\{L, d\}$. In the former case the channel is non-sparse, in the latter the error event is non-sparse. If the channel is non-sparse the diversity gain is determined by the minimum Hamming distance of the code and if it is larger or equal to d then full diversity is achieved.

Proof. The matrix $\tilde{\mathbf{F}}$ is a rectangular Vandermonde matrix with rank $\min\{L, d\}$.

A more general sufficient condition for $\tilde{\mathbf{F}}$ to be full rank is stated in the following proposition. We have found by numerical inspection that the condition is not necessary.

Proposition 7. Let $a \in \{1,..., N-1\}$ be relatively prime with N, i.e., $GCD\{N, a\} = 1$ then, assuming $L \le d$ the diversity gain equals L if

$$\max_{j=1,...,L} \{a\delta_j \bmod N\} < d \tag{15}$$

while assuming $L \ge d$ the diversity gain equals d if

$$\max_{i=1,...,d} \{a\tau_i \bmod N\} < L. \tag{16}$$

Proof. We prove (15) . Similarly, we can prove (16).

Let us assume $L \le d$. Then, $\tilde{\mathbf{F}}$ is not full rank iff there exists a non-zero L-tuple $\{\lambda_j\}$ such that $\sum_j \lambda_j \omega^{\tau_i \delta_j} = 0 \quad \forall\ i = 1,...d$. This is true iff there exists a polynomial $P(x) = \sum_j \lambda_j x^{\delta_j}$ that is zero for $x = \omega^{\tau_i}$ $\forall\ i = 1,...d$. Consequently, it can be written as the product of two polynomials, $P(x) = Q(x) \prod_{i=1}^{d} (x - \omega^{\tau_i})$. The right most polynomial has degree d, while $P(x)$ has degree $Deg\{P(x)\} \le \max_{j=1,...,L} \{\delta_j\}$. Therefore, a necessary condition for $\tilde{\mathbf{F}}$ to have rank lower than L is that $d \le \max_{j=1,...,L} \{\delta_j\}$. Consequently, a sufficient condition for $\tilde{\mathbf{F}}$ to have full rank L is that $d > \max_{j=1,...,L} \{\delta_j\}$. This proves (15) for $a = 1$.

Now, let $a \in \{1,..., N-1\}$ such that $GCD\{N, a\} = 1$. Then, a has the inverse, i.e., it exists $a^{-1} \in \{1,..., N-1\}$ such that $aa^{-1} \bmod N = 1$. Let us take $S(y) = P(y^a)$, then if $P(\omega^{\tau_i}) = 0$ for all $i = 1,..., d$, also $S(\omega^{a^{-1}\tau_i}) = P(\omega^{aa^{-1}\tau_i}) = P(\omega^{\tau_i}) = 0$ for all $i = 1,..., d$. Therefore, a necessary condition for the rank to be lower than L is that $d \le \max_{j=1,...,L} \{a\delta_j \bmod N\}$. Consequently, (15) is a sufficient condition for $\tilde{\mathbf{F}}$ to have full rank L.

Proposition 8. Let $\boldsymbol{\varepsilon}_L$ be an error sequence of length $L \geq d$ that achieves full diversity d, then every error sequence $\boldsymbol{\varepsilon}_{L'}$ of length $L' \geq L$ that has $\boldsymbol{\varepsilon}_L$ as sub-sequence achieves full diversity d. Further, $|\mathbf{G}_{L'}| > |\mathbf{G}_L|$ where $|\mathbf{G}_L|$ is the determinant of $\mathbf{G}$ corresponding to $\boldsymbol{\varepsilon}_L$. It can be computed recursively as follows:

$$|\mathbf{G}_{L+1}| = |\mathbf{G}_L|\,(1 + |\varepsilon^{k_{L+1}}|^2\, \mathbf{f}_{L+1}^H \mathbf{G}_L^{-1} \mathbf{f}_{L+1}) \qquad (17)$$

with the initial determinant equal to

$$|\mathbf{G}_d| = |\mathbf{F}_d|^2 \prod_{i=1}^{d} |\varepsilon^{k_i}|^2 . \qquad (18)$$

Proof. Let us consider $L' = d+1$, then we can write

$$\begin{aligned} \mathbf{G}_d = \mathbf{F}_d^H \mathbf{E}_d \mathbf{F}_d \quad \mathbf{G}_{d+1} &= [\mathbf{F}_d^H \vdots \mathbf{f}_{d+1}] diag\{\mathbf{E}_d, |\varepsilon^{k_{d+1}}|^2\} [\mathbf{F}_d^H \vdots \mathbf{f}_{d+1}]^H \\ &= \mathbf{F}_d^H \mathbf{E}_d \mathbf{F}_d + |\varepsilon^{k_{d+1}}|^2\, \mathbf{f}_{d+1} \mathbf{f}_{d+1}^H \end{aligned} \qquad (19)$$

with $\mathbf{f}_{d+1}$ being an appropriate $d \times 1$ vector that is associated to the error sequence $\boldsymbol{\varepsilon}_{d+1} = [\boldsymbol{\varepsilon}_d, |\varepsilon^{k_{d+1}}|^2]$. It follows that the determinant of $\mathbf{G}_{d+1}$ is

$$\begin{aligned} |\mathbf{G}_{d+1}| &= |\mathbf{F}_d^H \mathbf{E}_d \mathbf{F}_d|\,(1 + |\varepsilon^{k_{d+1}}|^2\, \mathbf{f}_{d+1}^H (\mathbf{F}_d^H \mathbf{E}_d \mathbf{F}_d)^{-1} \mathbf{f}_{d+1}) \\ &= |\mathbf{G}_d|\,(1 + |\varepsilon^{k_{d+1}}|^2\, \mathbf{f}_{d+1}^H \mathbf{G}_d^{-1} \mathbf{f}_{d+1}) > |\mathbf{G}_d| . \end{aligned} \qquad (20)$$

Note that the first equality (20) holds since $\mathbf{G}_d$ is assumed to be full rank. The last inequality holds since $\mathbf{G}_d$ is positive definite. Finally, $|\mathbf{G}_d| = |\mathbf{F}_d^H||\mathbf{E}_d||\mathbf{F}_d| = |\mathbf{F}_d^H|^2 |\mathbf{E}_d|$, which proves (18), while (17) is obtained recursively from (20).

Proposition 9. If the channel is non-sparse, $\mathbf{G}_d$ has determinant (21)

$$|\mathbf{G}_d| = \prod_{m=1}^{d} |\varepsilon^{k_m}|^2 \prod_{\substack{n,m=1\\n>m}}^{d} |\omega^{k_n} - \omega^{k_m}|^2 = \prod_{m=1}^{d} |\varepsilon^{k_m}|^2 \prod_{\substack{n,m=1\\n>m}}^{d} 4\sin^2\left(\frac{\pi}{N}(k_n - k_m)\right)$$

Proof. In such hypothesis $\mathbf{F}$ is a Vandermonde matrix of size d, and (21) follows from (18).

5. Distance and Complexity Constraints

Any block code (and also any finite length trellis codes) of rate R bits/symbol with codewords consisting of N_B blocks of length N symbols that belong to an *M*-ary alphabet satisfies the Singleton Bounds [4]: $d_{TH} \leq 1 + \lfloor N_B(1 - R/\log_2 M) \rfloor$ $d_{FH} \leq 1 + \lfloor N(1 - R\log_2 M) \rfloor$. Therefore, the diversity capability of the code is not influenced by the constellation shape. However, for a given rate, to increase the code diversity the size of the constellation has to increase. In terms of complexity, a trellis code can achieve diversity d only if its constraint length is at least $d-1$. If the code has rate R, then, to achieve diversity d its complexity (number of states in the trellis) has to be at least $2^{R(d-1)}$ [6]. Therefore, the complexity increases exponentially with the diversity order exploitation capability.

6. Interleaving

Interleaving is often deployed at the output of the channel encoder, and before the OFDM modulator [3],[7],[10]. Interleaving can be performed either in frequency or in time, or jointly in time and frequency. The most interesting case is when both coding and interleaving are deployed across one single OFDM symbol, or equivalently across a number of OFDM symbols over which the channel remains static. In such a case the only source of diversity is the frequency diversity provided by the dispersive channel. As we have shown in the previous section the diversity gain is a function not only of the Hamming distance of the code but it also depends on the position of the errors within the codeword. When we deploy an interleaver the pattern of the errors is clearly influenced. In order to investigate the effect we assume to deploy a uniform interleaver. The interleaver maps a given error pattern of Hamming weight d_{FH} and length N in all its possible distinct permutations with equal probability. The same analysis tool was used in [1] although in our case there is no code concatenation. Let $\boldsymbol{\varepsilon}$ be an error sequence of Hamming weight d_{FH}. Then, we can evaluate the average pairwise error probability computed over all possible interleavers $\Im$:

$$\overline{PEP}(\hat{\boldsymbol{\varepsilon}}) = \left[(d_{FH})! \binom{N}{d_{FH}} \right]^{-1} \sum_{\Im} P(\hat{\boldsymbol{\varepsilon}} \mid \hat{\boldsymbol{\varepsilon}} = \Im(\boldsymbol{\varepsilon})) . \qquad (22)$$

6.1 Effect of Interleaving on the Diversity Gain

Assuming a quasi-static channel the interleaver has the effect of changing the error patterns (positions). If the channel is not sparse then the interleaver has no effect on diversity (see Proposition 6). On the contrary if the channel is sparse then a given full diversity error event may be mapped by the interleaver into a lower diversity error event, or vice versa, into a higher diversity error event. Overall, the effect of interleaver is to average out the diversity benefit and allow for simpler "although heuristic" code design. That is, it makes the probability of the code to achieve full diversity high.

6.2 Effect of Interleaving on the Coding Gain

Again we assume a quasi-static channel. If we assume the channel to be non-sparse, interleaving does not affect diversity however it changes the error patterns such that the coding gain may change. As an example, if we pick a maximum diversity BPSK code, the minimum coding gain is achieved in correspondence to the error sequence $\boldsymbol{\varepsilon}_d$ with Hamming weight d that minimizes (21). Further, the average pairwise error probability can be bounded as

$$\overline{PEP} \leq \frac{1}{2}\left[\binom{N}{d} |R| \left(\frac{E_S}{N_0}\right)^d \right]^{-1} \sum_{\mathbf{k}\in\Im} \prod_{\substack{n,m=1\\n>m}}^{d} \left(4\sin^2 \frac{\pi}{N}(k_n - k_m)\right)^{-1} \qquad (23)$$

where $\Im$ is the set of all possible combinations of the error position indices in the *d*-tuple $\mathbf{k} = [k_1, ..., k_d]$ with $0 \leq k_i < N-1$.

7. Examples

7.1 Two-Taps Channel

Let us consider a 2-ray channel model with uncorrelated channel taps, i.e., $\mathbf{R} = diag\{[P_1,\ P_2]\}$, $p_1 = 0$, $p_2 \geq 1$. Then,

$$\mathbf{RG} = \begin{bmatrix} P_1 \sum_{i=1}^{L} |\varepsilon^{k_i}|^2 & P_1 \sum_{i=1}^{L} |\varepsilon^{k_i}|^2\, e^{-j\frac{2\pi}{N}k_i p_2} \\ P_2 \sum_{i=1}^{L} |\varepsilon^{k_i}|^2\, e^{j\frac{2\pi}{N}k_i p_2} & P_2 \sum_{i=1}^{L} |\varepsilon^{k_i}|^2 \end{bmatrix} \qquad (24)$$

and compute in closed form the determinant and the eigenvalues,

$$|\mathbf{RG}| = \lambda_1 \lambda_2 = 4P_1P_2 \sum_i \sum_{j>i} |\varepsilon^{k_i}|^2 |\varepsilon^{k_j}|^2 \sin^2\left(\frac{\pi}{N} p_2 (k_j - k_i)\right) \qquad (25)$$

$$\lambda_{1,2} = \frac{1}{2}(P_1+P_2)\sum_i |\varepsilon^{k_i}|^2 \pm \frac{1}{2}\left[\left((P_1+P_2)\sum_i |\varepsilon^{k_i}|^2\right)^2 - 16P_1P_2\sum_i\sum_{j>i}|\varepsilon^{k_i}|^2|\varepsilon^{k_j}|^2 \sin^2\left(p_2(k_j-k_i)\pi/N\right)\right]^{1/2} \quad (26)$$

Now, for all error sequences for which $p_2(k_j - k_i) \bmod N = 0$ the diversity order equals one, and the only non-zero eigenvalue is $\lambda_1 = (P_1+P_2)\sum_i |\varepsilon^{k_i}|^2$. For instance, let's pick a geometrically uniform code that has a single minimum Hamming distance codeword that is equal to two. The encoder is followed by uniform interleaving. If N/p_2 is integer, and if the distance among the two errors is equal to $\tau_2 = N/p_2$ the error event does not achieve full diversity. In the left plot of Fig. 1 we show the probability to achieve full diversity as a function of the tap delay p_2. It has been computed assuming all distinct pairs k_1, k_2 with equal probability. In the right plot of Fig. 1 we show the distribution of the normalized coding gain assuming BPSK mapping. As the figures show, the probability of achieving full diversity increases when the number of tones (interleaver length) increases. The coding gain distribution is a function of the number of tones (interleaver length).

7.2 Repetition Code

Let us consider a repetition code with rate $R = 1/L$. It generates for each input symbol a codeword of length equal to L, thus, the minimum Hamming distance equals L. Let us assume a channel with rank $d \geq L$. The frequency diversity gain depends on how we map the codewords into the sub-channels. It certainly equals L if we transmit the $N = LK$ coded symbols in the natural order across the N tones. In fact in such a case Proposition 6 holds. Instead if we deploy a uniform interleaver, there is a non-zero probability that the interleaved codewords generate error patterns for which the achieved diversity is less than L. If the channel is not sparse then diversity L is achieved for all interleaver realizations. Further, the coding gain distribution changes resulting in possibly better average pairwise error probability. This is shown in Fig. 2 where we consider a 4 taps non-sparse channel with $\mathbf{R} = 2.5^{-1/2} diag\{[1, 0.75, 0.5, 0.25]\}$, and we compute according to (9) the pairwise error probabilities corresponding to all possible interleaver patterns. We assume an error event with weight 2 or 4, with BPSK mapping. It is interesting to note that the worst performance (lowest coding gain) is achieved by a non-sparse error pattern. As the interleaver length (number of tones *N*) increases the PEP spectrum broadens. In any case there exist interleavers that significantly better the PEP performance.

7.3 Trellis Code

If we consider trellis codes, the performance is determined not only by the Hamming distance of the code but also by the error positions. The construction of trellis codes has to be carried out by searching for codes for which $\tilde{\mathbf{F}}$ is full rank. A widely deployed approach is to use binary convolutional codes followed by interleaving and symbol mapping. According to our analysis the effect of interleaving is to average out performance, i.e, to allow for maximum diversity and coding gains with high probability.

8. Conclusions

Several necessary and sufficient conditions have been derived for achieving maximum diversity and coding gains in COFDM. The effect of interleaving on diversity and coding gain has been analytically studied.

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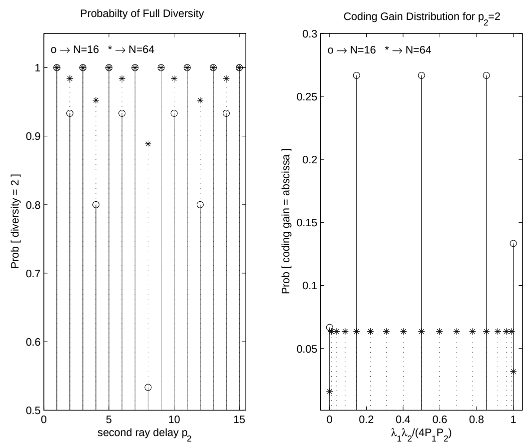

Fig.1. Probability of achieving full diversity, and coding gain distribution.

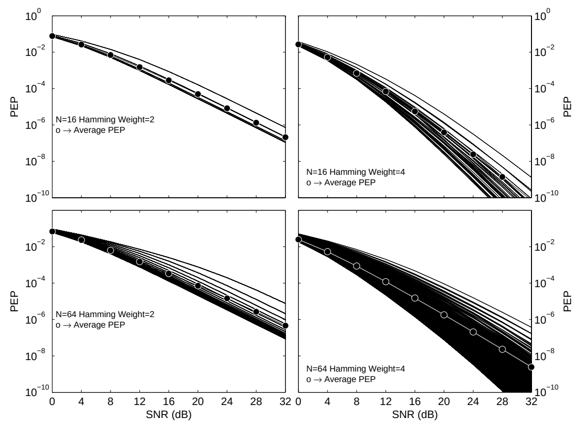

Fig.2. Spectrum of pairwise error probability with 4 taps fading channel.